Dissipative perturbation theory for quantum fields

LETTER TO THE EDITOR

Dissipative perturbation theory for quantum fields

Ian D Lawrie

Department of Physics, University of Leeds, Leeds LS2 9JT, UK

Abstract. A perturbative method is proposed for studying the evolution in time of quantum field theories which are driven away from thermal equilibrium by the presence of explicitly time-dependent parameters in their Hamiltonians. For the case of a scalar field with time-dependent mass, it is shown within the closed-time-path formalism that absorptive parts of higher-order contributions to propagators can be approximately resummed by the addition and subtraction of a local but non-diagonal quadratic counterterm. In this way, lowest-order propagators are obtained which mimic the dissipative behaviour of the complete propagators and thereby facilitate the incorporation of finite relaxation times in low-order calculations. Under highly favourable conditions, a Boltzmann equation can be approximately derived.

Several years ago, Semenoff and Weiss (1985a, b) used a path integral method to derive Feynman rules for the perturbative evaluation of Green functions in non-equilibrium scalar field theories. Their analysis uses the closed-time-path formalism which derives ultimately from the work of Schwinger (1961) and Keldysh (1965). Given a state of thermal equilibrium at an initial time t_0, it allows for explicit time dependence of one or more parameters in the theory, which in general will drive the system away from equilibrium at later times. In principle, therefore, it should be possible to follow the evolution in time of the non-equilibrium density matrix, or at any rate of its moments which are the Green functions. The physical problem which motivated the work of Semenoff and Weiss, as well as that reported here, is the dynamics of phase transitions in the early universe, for which the basic machinery is that of relativistic quantum field theory. Clearly, however, analogous situations abound in condensed matter physics, and the modified formalism constructed here should be readily adaptable to non-relativistic statistical mechanics.

While the Feynman rules derived by Semenoff and Weiss are formally correct, they are virtually useless in practice, because they do not incorporate the effects of dissipation and relaxation. The reason for this is clear. Perturbation theory formulated in the normal way is an expansion about a non-interacting theory. Only when interactions occur, however, can the occupation numbers of single-particle or quasiparticle modes evolve with time in the expected manner. In the interacting theory, one may expect dissipation to emerge from absorptive parts of the full Green functions, but these can be adequately represented in perturbation theory only by resumming an infinite series of diagrams. For a time-independent equilibrium state, for example, one expects energy denominators of the form $(\omega \pm \omega_k \pm i\gamma_k)$, where the decay width γ_k for quasiparticle excitations of momentum k is typically of the order of the square of a coupling constant. In order to isolate an expression of this kind, one must identify an appropriate geometric series within the perturbative expansion. In a non-equilibrium theory with explicit

time dependence, one can contemplate calculating only a very few low-order terms and, moreover, no energy (or frequency) representation is possible. It is obviously essential to cast perturbation theory in a form in which the unperturbed propagators mimic as closely as possible the incalculable full propagators. What we propose here amounts to a renormalisation scheme wherein the desired resummation is effected by adding an appropriate local counterterm to the free-field part of the Lagrangian and subtracting it from the interaction part. Adjustable functions in the counterterm can be chosen self-consistently so as to minimise the net effect of the modified interaction.

To be specific, we consider a self-interacting Hermitian scalar field, with Lagrangian density

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\partial_t\phi)^2 - \tfrac{1}{2}|\nabla\phi|^2 - \tfrac{1}{2}m^2(t)\phi^2 - (\lambda/4!)\phi^4 \tag{1}$$

the mass carrying an explicit time dependence which we take to be externally prescribed. For example, the Lagrangian density of a scalar field in a spatially flat Robertson-Walker universe may be expressed in this form, if t is identified as conformal time and the spatial coordinates are comoving. The generating functional for time-ordered Green functions is

$$Z(j) = \mathrm{Tr}\left[\exp(-\beta_0 H(t_0))\, T\exp\left(\int_{t_0}^{\infty} dt \int d^3x\, j(x, t)\phi(x, t)\right)\right] \tag{2}$$

where $H(t_0)$ is the Hamiltonian obtained in the canonical manner from (1), evaluated at the initial time t_0, and β_0 is the inverse of the initial temperature. Its path integral representation involves three c-number fields, $\phi_1(x, t)$, $\phi_2(x, t)$, $\phi_3(x, \tau)$, which live on three segments of a contour in the complex time plane: C_1 runs along the real axis from t_0 to ∞, C_2 returns from ∞ to t_0 and C_3 descends from t_0 to $t_0 - i\beta_0$. Imposition of causal boundary conditions is equivalent to giving the horizontal segments an infinitesimal downward slope. The action which appears in the functional integral is given by

$$iS = \int d^3x\left(i\int_{t_0}^{\infty} dt[\mathcal{L}(\phi_1) - \mathcal{L}(\phi_2)] - \int_0^{\beta_0} d\tau\, \hat{\mathcal{L}}(\phi_3)\right) \tag{3}$$

with

$$\hat{\mathcal{L}}(\phi) = \tfrac{1}{2}(\partial_\tau\phi)^2 + \tfrac{1}{2}|\nabla\phi|^2 + \tfrac{1}{2}m^2(t_0)\phi^2 + (\lambda/4!)\phi^4. $$

We restrict our attention to the real-time propagators $G_{ab}(x_1, t_1; x_2, t_2) = \langle\phi_a(x_1, t_1)\phi_b(x_2, t_2)\rangle$ for which a and b are either 1 or 2, where $\langle\ldots\rangle$ denotes the functional average weighted with $\exp(iS)$. In terms of the original quantum field, we may identify

$$\begin{pmatrix} G_{11}(1; 2) & G_{12}(1;2) \\ G_{21}(1; 2) & G_{22}(1; 2) \end{pmatrix} = \mathrm{Tr}\left[\exp(-\beta_0 H(t_0))\begin{pmatrix} T[\phi(1)\phi(2)] & [\phi(2)\phi(1)] \\ [\phi(1)\phi(2)] & \bar{T}[\phi(1)\phi(2)] \end{pmatrix}\right] \tag{4}$$

where T and $\bar{T}$ are respectively, the time- and anti-time-ordering operators and '1' and '2' are an obvious shorthand for the spacetime arguments (see, e.g., Semenoff and Weiss (1985a) or Chou *et al* (1985)). From this identification, it is straightforward to show that the propagator matrix has the structure

$$G(t; t') = \theta(t - t')\begin{pmatrix} H(t, t') & H^*(t, t') \\ H(t, t') & H^*(t, t') \end{pmatrix} + \theta(t' - t)\begin{pmatrix} H(t', t) & H(t', t) \\ H^*(t', t) & H^*(t', t) \end{pmatrix} \tag{5}$$

where $H(t, t')$ is an unknown complex function and spatial arguments have been suppressed.

We look for an unperturbed propagator matrix $g(t; t')$ which has the same structure as (5). Since we have taken a spatially homogeneous system, we may take a Fourier transform with respect to the spatial argument $(x - x')$. The transformed propagator should satisfy

$$\mathscr{D}_k(t, \partial/\partial t)g_k(t, t') = g_k(t, t')\mathscr{D}_k(t', -\bar{\partial}/\partial t') = -i\delta(t - t') \tag{6}$$

where we take $\mathscr{D}_k$ to be of second order in the time derivative. We find that the most general operator consistent with the causal structure reflected in (5) is

$$\mathscr{D}_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{\partial^2}{\partial t^2} + \beta_k(t)\right) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\gamma_k(t)\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{\gamma}_k(t)\right) + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}(-i\alpha_k(t)) \tag{7}$$

where $\alpha_k(t)$, $\beta_k(t)$ and $\gamma_k(t)$ are arbitrary real functions. In standard perturbation theory, the quadratic part of (1) leads to the identification $\beta_k(t) = k^2 + m^2(t)$ and $\alpha_k(t) = \gamma_k(t) = 0$. To incorporate dissipative effects into $g_k(t, t')$, we allow more general functions by adding and subtracting an appropriate quadratic counterterm in (3). This counterterm is clearly not diagonal in the fields ϕ_1 and ϕ_2 and thus has no simple interpretation in terms of the original quantum field and Lagrangian (1), except that $\beta_k(t)$ includes mass and wavefunction renormalisations.

To carry out perturbative calculations, we must first solve (6) for g_k in terms of α_k, β_k and γ_k, and then determine these functions self-consistently by imposing suitable generalised renormalisation conditions which match higher-order contributions to G which arise from the counterterm against those arising from interactions. That such matching can be carried out is guaranteed by the fact that we have chosen the operator (7) for consistency with (5). We solve (6) for a function $h_k(t, t')$ which will be the lowest-order approximation to $H_k(t, t')$ in the Fourier transform of (5). For orientation, consider first the case in which α_k, β_k and γ_k are constant, corresponding to a stationary state with constant m. The solution is

$$h_k(t, t') = \tfrac{1}{4}\exp[-i(\omega_k - \tfrac{1}{2}i\gamma_k)(t - t')]\left(\frac{1}{\omega_k} + \frac{2n_k + 1}{(\omega_k - \tfrac{1}{2}i\gamma_k)}\right)$$

$$+ \tfrac{1}{4}\exp[i(\omega_k + \tfrac{1}{2}i\gamma_k)(t - t')]\left(-\frac{1}{\omega_k} + \frac{2n_k + 1}{(\omega_k + \tfrac{1}{2}i\gamma_k)}\right) \tag{8}$$

where $\omega_k = (\beta_k - \gamma_k^2/4)^{1/2}$ and $(2n_k + 1) = \alpha_k/\omega_k\gamma_k$. The interpretation of n_k as the occupation number of quasiparticle modes and γ_k as the decay width is clear. For the time-dependent case, we obtain a partial solution in terms of auxiliary functions $\Omega_k(t)$, $C_k(t, t')$ and $N_k(t, t')$ with C_k real. Quite generally, we may write

$$h_k(t, t') = \exp\left(-i\int_{t'}^{t}[\Omega_k(t'') - \tfrac{1}{2}i\gamma_k(t'')]\,dt''\right)[C_k(t, t') + N_k(t, t')]$$

$$+ \exp\left(i\int_{t'}^{t}[\Omega_k(t'') + \tfrac{1}{2}i\gamma_k(t'')]\,dt''\right)[-C_k(t, t') + N_k^*(t, t')]. \tag{9}$$

The desired solution is then given by

$$C_k(t, t') = \tfrac{1}{4}[\Omega_k(t)\Omega_k(t')]^{-1/2} \tag{10}$$

and

$$N_k(t, t') = C_k(t, t')\bar{N}_k(t') \tag{11}$$

provided that $\Omega_k(t)$ and $\bar{N}_k(t)$ satisfy

$$\frac{1}{2}\frac{\ddot{\Omega}_k(t)}{\Omega_k(t)} - \frac{3}{4}\frac{\dot{\Omega}_k^2(t)}{\Omega_k^2(t)} + \Omega_k^2(t) = \beta_k(t) - \tfrac{1}{4}\gamma_k^2(t) \tag{12}$$

and

$$\left(\frac{\partial}{\partial t} + \gamma_k(t) + 2\mathrm{i}\Omega_k(t) - \frac{\dot{\Omega}_k(t)}{\Omega_k(t)}\right)\left(\frac{\partial}{\partial t} + \gamma_k(t)\right)\bar{N}_k(t) = 2\mathrm{i}\alpha_k(t). \tag{13}$$

We stress that this is quite general: to the author, at least, it came as a pleasant surprise that the solution can be thus decomposed into functions with a single time argument.

Further progress requires a detailed prescription for obtaining $\alpha_k(t)$, $\beta_k(t)$ and $\gamma_k(t)$, together with suitable approximations for solving (12) and (13). It also requires boundary conditions for these equations. In principle, boundary conditions arise from continuity of the full 3×3 matrix of propagators along the complex time contour and periodic boundary conditions at its endpoints (see Semenoff and Weiss 1985a). However, the lowest-order terms of perturbation theory do not couple the real and imaginary time propagators. In practice, therefore, it is probably an adequate approximation to circumvent a complete analysis by supplying initial conditions from physical intuition, requiring, for example, that $\bar{N}_k(t)$ reproduce the Bose occupation numbers in an initial equilibrium state.

As a qualitative illustration of the practical application of the formalism developed here, suppose that (i) all functions are slowly varying in time and (ii) $\gamma_k(t)$, which is of order λ^2 in the theory defined by (1), is sufficiently small for a quasiparticle mass shell to be approximately defined. Under these conditions, we have been able to show that $\bar{N}_k(t)$ can be interpreted in terms of a time-dependent set of occupation numbers $n_k(t)$ and that (13) reduces to a Boltzmann equation. The scattering term in this equation arises from matching the counterterm containing α_k and γ_k to the diagram shown in figure 1. In principle, the expectation value of any operator can be calculated at any time by integrating this equation and (12), by numerical methods if necessary. This is, of course, the simplest conceivable approximation. By making it, we allow our equations to degenerate into a form which might well have been guessed at by a variety of informal kinetic theory arguments. Nevertheless, we believe that the formalism presented here provides a systematic means of improving upon such elementary guesswork. Details of our investigation, together with illustrative numerical calculations will be described in a forthcoming longer paper.
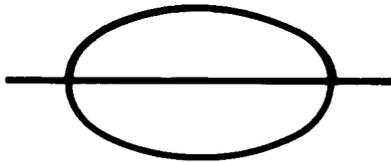

Figure 1. Simplest contribution to propagators having an absorptive part.

Finally, it is appropriate to mention two other approaches to non-equilibrium field theory which are superficially similar to ours. Calzetta and Hu (1986) have applied the closed-time-path method to a field theory with constant parameters, but with a general Gaussian initial density matrix. They derive an approximate Boltzmann equation by truncating the infinite hierarchy of Dyson–Schwinger equations. We believe that the technique proposed here is both more systematic and more versatile

than theirs, though at the level of approximation needed to derive the Boltzmann equations they become more or less equivalent. Umezawa and his collaborators have recently extended the formalism known as thermofield dynamics to encompass non-equilibrium states (see, e.g., Arimitsu *et al* (1988) and Umezawa and Yamanaka (1987)). This theory is based on an axiomatic extension of standard quantum field theory and shares with the closed-time-path formalism a doubling (though not a tripling) of field degrees of freedom to accommodate thermal states. For equilibrium states, it turns out to be equivalent to quantum statistical mechanics as exemplified by the trace formula (2). We strongly suspect that this equivalence does not extend to non-equilibrium states, but cannot demonstrate this conclusively at present. At any rate, the propagators obtained by Umezawa and Yamanaka for relativistic scalar fields differ significantly from ours, though a direct comparison may not be appropriate.

References

Arimitsu T, Guida M and Umezawa H 1988 *Physica* **148A** 1
Chou K-C, Su Z-B, Hao B-L and Yu L 1985 *Phys. Rep. C* **118** 1
Calzetta E and Hu B-L 1988 *Phys. Rev.* D **37** 2878
Keldysh L V 1965 *Sov. Phys.-JETP* **20** 1018
Schwinger J 1961 *J. Math. Phys.* **2** 407
Semenoff G W and Weiss N 1985a *Phys. Rev.* D **31** 689
—— 1985b *Phys. Rev.* D **31** 699
Umezawa H and Yamanaka Y 1987 *Time dependent non-equilibrium thermo field dynamics of type-2 fields.*
    Preprint University of Alberta